Optimization and Computational Linear Algebra for Data Science Midterm

October 22, 2019

- **The exam ends at the end of the time of the lecture.**
- **Please justify your answers, proving the statements you make. You are allowed to refer to results shown in lectures/recitations/homeworks as long as you state them precisely, meaning that you should say exactly which hypothesis are needed in the result you use.**
- **This exam is open book/notes. You are allowed to consult notes and books you bring, but not allowed to use electronic devices.**
- **If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, for example), state explicitly that you have done so. Solutions where extra conditions were assumed, or where only special cases where treated, will also be graded (probably scored as a partial answer).**
- **The exam has 2 pages. It has 7 question groups that together total 100 points plus extra credit ($\star$). Extra credit points will be added to your total score.**
- **If you have questions (or find a typo) let me know. Any eventual typo will be announced on the board.**

Problem 1 (10 points). *Let $A \in \mathbb{R}^{n\times n}$ be a matrix and $y \in \mathbb{R}^n$ be a non-zero vector. Are the following sets subspaces of $\mathbb{R}^n$? Justify your answer.*

(a) $E_1 = \{x \in \mathbb{R}^n \,|\, Ax = 3x\}$.

(b) $E_2 = \{x \in \mathbb{R}^n \,|\, Ax = y\}$.

Problem 2 (15 points). *Let $A \in \mathbb{R}^{n\times m}$, with $m \neq n$. Determine which of the following statements are equivalent.* ***No proof is required here****, simply write something like*

« Statements **(?),(?),(?),(?)** are equivalent to each other, and statements **(?),(?),(?),(?)** are equivalent to each other.»

(a) *The columns of A are linearly independent.*

(b) *The rows of A are linearly independent.*

(c) *The equation $Ax = 0$ has exactly one solution $x \in \mathbb{R}^m$.*

(d) $\mathrm{Ker}(A) = \{0\}$.

(e) $\mathrm{rank}(A) = n$.

(f) $\mathrm{rank}(A) = m$.

(g) *The span of the columns of A is $\mathbb{R}^n$.*

(h) $\mathrm{Im}(A) = \mathbb{R}^n$.

Problem 3 (10 points). ***True or false?*** *For each of the following, give a proof or find a counterexample.*

(a) *There are no 4×3 matrices with rank equal to 3.*

(b) *For all $A, B \in \mathbb{R}^{n\times n}$, $\mathrm{rank}(A + B) = \mathrm{rank}(A) + \mathrm{rank}(B)$.*

(c) *If a vector $x \in \mathbb{R}^n$ is orthogonal to every vector of $\mathbb{R}^n$ then $x = 0$.*

(**d**) *For all* $A \in \mathbb{R}^{n\times n}$*, if* $v_1 \in \mathbb{R}^n$ *is an eigenvector of* A *associated with the eigenvalue* λ_1 *and if* $v_2 \in \mathbb{R}^n$ *is an eigenvector of* A *associated with the eigenvalue* $\lambda_2 \neq \lambda_1$*, then* $v_1 + v_2$ *is an eigenvector of* A *associated with the eigenvalue* $\lambda_1 + \lambda_2$.

Problem 4 (10 points)**.** *Let* S *be a subspace of* $\mathbb{R}^n$ *and* P_S *be the orthogonal projection onto* S*. Show that for all* $x \in \mathbb{R}^n$,

$$\|x\|^2 = \|P_S x\|^2 + \|x - P_S x\|^2.$$

Problem 5 (15 points)**.** *Let* $M \in \mathbb{R}^{n\times n}$ *be a symmetric matrix such that* $M^{2019} = 0$*. Show that* $M = 0$.

Problem 6 (10 points)**.** *Let* $A \in \mathbb{R}^{n\times m}$*. Show that* $AA^\top$ *is invertible if and only is* $\mathrm{rank}(A) = n$.

Problem 7 (30 points)**.** *Let* $A \in \mathbb{R}^{n\times m}$ *and* $y \in \mathbb{R}^n$*. We assume that* $n < m$*. We let* $x^\star$ *be a minimizer of the function:*

$$f(x) = \|Ax - y\|.$$

(**a**) *Is* $x^\star$ *the unique minimizer of* f *? That is, does there exists* $x \in \mathbb{R}^m$ *such that* $x \neq x^\star$ *and* $f(x) = f(x^\star)$ *?*

(**b**) *Show that* $Ax^\star = P_{\mathrm{Im}(A)} y$*, where* $P_{\mathrm{Im}(A)}$ *denotes the orthogonal projection on* $\mathrm{Im}(A)$.

(**c**) *Deduce that for all* $x \in \mathbb{R}^m$,

$$\langle Ax, y - Ax^\star \rangle = 0.$$

(**d**) *Show that this implies that*

$$A^\top A x^\star = A^\top y.$$

(**e**) *Is the matrix* $A^\top A$ *invertible?*

Problem 8 (($\star$) 5 points)**.** *Let* $M \in \mathbb{R}^{n\times n}$ *be a symmetric matrix whose eigenvalues are strictly positive. Show that for all* $u, v \in \mathbb{R}^n$:

$$\big(u^\top M v\big)^2 \leq \big(u^\top M u\big)\big(v^\top M v\big).$$

Problem 9 (($\star$) 5 points)**.** *Let* $A, B \in \mathbb{R}^{n\times n}$ *two symmetric matrices that verify* $AB = BA$*. Show that there exists an orthogonal matrix* P *and two diagonal matrices (i.e. matrices with zeros outside of the diagonal)* D, D' *such that*

$$A = PDP^\top \qquad \text{and} \qquad B = PD'P^\top.$$