Optimization and Computational Linear Algebra for Data Science - Midterm

October 29, 2020

- **You have 1 hour and 40 minutes to work on this midterm exam, plus 20 min to scan and upload your work. In case you have troubles when uploading, email me your work at `lm4271@nyu.edu`. Late submissions (after the 2h) will not be accepted.**
- **Please justify your answers, proving the statements you make. You are allowed to refer to results shown in lectures/recitations/homeworks as long as you state them precisely, meaning that you should say exactly which hypothesis are needed in the result you use.**
- **This exam is open book/notes. You are allowed to consult notes and books you bring, but not allowed to use internet or to communicate with anyone.**
- **If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, for example), state explicitly that you have done so. Solutions where extra conditions were assumed, or where only special cases where treated, will also be graded (probably scored as a partial answer).**
- **The exam has 2 pages. It has 7 question groups that together total 100 points plus extra credit ($\star$). Extra credit points will be added to your total score (but grades will not exceed 100: points above 100 will be discarded).**

Important: In all the following problems, $\langle \cdot, \cdot \rangle$ denotes the Euclidean dot product and $\|\cdot\|$ denotes the Euclidean norm.

Problem 1 (10 points)**.** Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $v \in \mathbb{R}^n$ be a vector. Are the following sets subspaces of $\mathbb{R}^n$? Justify your answer.

(a) $E_1 = \left\{ x \in \mathbb{R}^n \,\middle|\, Ax = 3e_1 \right\}$.

(b) $E_2 = \left\{ x \in \mathbb{R}^n \,\middle|\, \langle v, x \rangle = 0 \right\}$.

Problem 2 (20 points)**. True or false?** For each of the following, give a proof (if you think the statement is true) or give a counterexample (if you think the statement is false).

(a) For all $A, B \in \mathbb{R}^{3\times3}$, if $\mathrm{rank}(A) \leq \mathrm{rank}(B)$ then $\mathrm{Im}(A) \subset \mathrm{Im}(B)$.

(b) Let $A \in \mathbb{R}^{2\times3}$ and $y \in \mathbb{R}^2$. If u, v are both solutions of the linear system $Ax = y$ then $(u - v) \in \mathrm{Ker}(A)$.

(c) There are no matrices $A \in \mathbb{R}^{4\times4}$, $B \in \mathbb{R}^{4\times3}$ and $C \in \mathbb{R}^{3\times4}$ such that $ABC = \mathrm{Id}_4$.

(d) If $A \in \mathbb{R}^{3\times3}$ is a matrix such that the sum of the coefficients of each row is equal to 2 (that is $A_{i,1} + A_{i,2} + A_{i,3} = 2$ for all $i \in \{1, 2, 3\}$), then 2 is an eigenvalues of A.

Problem 3 (12 points)**.** Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $(v_1, \dots, v_n)$ be an orthonormal family of eigenvectors of A, associated respectively to the eigenvalues $\lambda_1, \dots, \lambda_n$ (that is, $Av_i = \lambda_i v_i$ for all $i \in \{1, \dots, n\}$). We define

$$x = v_1 + \cdots + v_n.$$

Compute (and justify your answer) $\|x\|^2$ and $\|Ax\|^2$ in terms of n and the eigenvalues of A.

Problem 4 (14 points)**.** Let $u, v \in \mathbb{R}^n$.

(**a**) In this question, we assume that for all $x \in \mathbb{R}^n$ we have $\langle u, x \rangle = \langle v, x \rangle$. Show that $u = v$.

(**b**) In this question, we consider a subspace S of $\mathbb{R}^n$ and denote by P_S the orthogonal projection onto S. We assume that for all $x \in S$ we have $\langle u, x \rangle = \langle v, x \rangle$. Show that $P_S(u) = P_S(v)$.

Problem 5 (12 points)**.** Let $A \in \mathbb{R}^{n \times r}$ and $B \in \mathbb{R}^{r \times n}$ such that $\mathrm{rank}(A) = \mathrm{rank}(B) = r$. Show that $\mathrm{rank}(AB) = r$

Problem 6 (20 points)**.** Let $M \in \mathbb{R}^{n \times m}$ be a matrix whose columns are linearly independent. Let $w \in \mathbb{R}^n$ and let $u \in \mathbb{R}^n$ be the orthogonal projection of w onto $\mathrm{Im}(M)$

(**a**) Show that $M^\top M$ is invertible.

(**b**) Show that for all $x \in \mathbb{R}^m$, $(Mx) \perp (w - u)$.

(**c**) Deduce that $M^\top u = M^\top w$

(**d**) Conclude that

$$u = M(M^\top M)^{-1} M^\top w.$$

Hint: use the fact that $u \in \mathrm{Im}(M)$.

Problem 7 (12 points)**.** Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix such that $M^{2020} = \mathrm{Id}_n$. Compute M^2 (you are of course asked to justify your answer).

Problem 8 (($\star$) 5 points)**.** Let $A \in \mathbb{R}^{10 \times 10}$ such that $A^2 = 0$. Show that $\mathrm{rank}(A) \leq 5$.